Random matrices and integrable systems: introduction and overview

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PREFACE

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In June–July 2005, a three-week meeting took place at the Centre de Recherches Mathématiques in Montréal (CRM), on the topic 'Random matrices, random processes and integrable systems'. There were a total of 75 participants from 18 countries around the world. Nearly half were invited speakers; the rest were mainly younger researchers, postdoctoral fellows and graduate students, many of whom also presented talks on their work. This extended event included eight daily lecture series of one week's duration, given by a number of top experts in the field, which will be published as a separate monograph, as well as afternoon workshop sessions reporting on current work.

All participants, as well as a number of other researchers active in the field, were invited to contribute papers to this special issue which focusses on the areas covered by the meeting. Very few of the papers in this issue actually coincide with those presented at the workshop, but they all focus on the same set of topics.

The area of random matrices, stimulated greatly by numerous potential applications in physics, as well as by the study of related problems in probability and combinatorics, has enjoyed a rather remarkable period of progress in recent years. Methods originating in the theory of integrable systems, such as the Riemann–Hilbert approach to inverse spectral problems, have played a surprisingly important role in many of the more recent developments. Connections have also been made with several other problems of a probabilistic nature that are amenable to analysis by similar methods. This issue and the CRM meeting that inspired it represent a good sample of the very diverse and fascinating methods, results and applications emerging from this field of study.

The fact that there exist close relations between problems of a probabilistic nature and integrable systems may at first seem puzzling since the latter are anything but random. The apparent paradox of their cohabitation is partly resolved if one notes that it is very useful to approach the study of probabilistic systems not just by restricting them to a fixed probability measure, but embedding the latter into parametric families and studying the effects of deformations. This naturally leads to differential and difference equations governing such deformations. Asymptotic studies with respect to some limiting parameter, such as the matrix dimension, or the endpoints of spectral gaps also give rise to differential or difference equations with respect to these parameters. The remarkable fact that arises is that these equations turn out typically to be exactly of the type long studied in the theory of integrable systems through inverse spectral methods.

This volume has been grouped rather roughly into three parts. The first mainly concerns results on random $N \times N$ matrices, for finite N, together with related tools, such as the theory of orthogonal polynomials. The second part mainly involves asymptotic results as $N \to \infty$, and the third concerns topics in which related considerations arise, either in a probabilistic setting, or in the area of integrable systems. The grouping is necessarily somewhat artificial, since more than one of these elements may be present in a single work, but it has been done to help give some structure to the sequence of papers included.

The first part consists of papers by M C Bergere (p8749), S Ghosh (p8775), J Harnad and A Yu Orlov (p8783) and I Nenciu (p8811). M C Bergere's paper is a detailed account of integration methods used to evaluate correlation functions of spectral invariants for the complex-matrix model, with emphasis on the Gaussian case and its applications to the Berenstein–Maldacena–Nastase (BMN) limit of super Yang–Mills theory. While the application is for large N, the methods used in this work provide results valid for finite values of N. S Ghosh derives a Christoffel–Darboux identity for skew–symmetric polynomials and polynomial potentials. This is an important step towards a systematic study of algebraic and analytic properties of these polynomials, which appear in the orthogonal and symplectic ensembles. J Harnad and A Yu Orlov develop a new method based upon standard determinantal identities and multivariable partial fraction expansions, for evaluating certain integrals of rational symmetric functions that appear in the evaluation of correlators of characteristic polynomials in two-matrix models. I Nenciu provides an overview regarding the theory and applications of Cantero–Moral–Velázquez (CMV) matrices. These replace the more familiar tridiagonal Jacobi matrices that appear in the theory of orthogonal polynomials on the line when the measure is on the unit circle and the orthogonalizing sequence is obtained from Gram-Schmidt's method starting from the sequence of alternating powers $1, z, 1/z, z^2, z^{-2}, \ldots$. The spectral theory of CMV matrices allows a treatment of arbitrary β ensembles with certain external potentials and connects closely to the Ablowitz–Ladik lattice and its integrability.

The second part of this special issue, relating mainly to large N asymptotics, consists of papers by M Bertola (p8823), L Chekhov (p8857), A Borodin and A Novikov (p8895), T Grava (p8905), R Teodorecu (p8921) and A Zabrodin and P Wiegmann (p8933).

M Bertola explores the formal properties of the free energy for the two-matrix model in the presence of 'semiclassical' potentials (i.e. those with rational derivatives) in the planar limit. The role of boundaries for the spectra (hard edges) is explained from the algebro-geometric point of view in terms of the asymptotic spectral curve of the model. The deformations lead to an extended Whitham hierarchy. L Chekhov's paper also concerns formal properties, in the large N limit, of the one-matrix model with hard-edges. The full perturbative expansion in inverse powers of N is constructed, using rather advanced tools of algebraic geometry. The work of A Borodin and A Novikov concerns the neighbouring topic of the spectral theory of Töplitz operators. They give a self-contained proof of a conjecture of H Widom on the positivity of the spectrum of the Töplitz matrix associated with a certain class of *symbols* (i.e. meromorphic functions). Connections to the Schur process and related correlation functions are explained. T Grava ties together certain of the formal investigations already appearing in the literature for the one-matrix-model in the multi-cut regime and the rigorous Riemann–Hilbert analysis developed in recent years, putting on a firmer footing the formal manipulations leading to the computation of higher-order partial derivatives of the partition function. R Teodorescu investigates the connections between the normal matrix model and conformal geometry of domains in the complex plane. When the external harmonic potential is fine-tuned the domain in which the eigenvalues cluster develops singularities on its boundary, which signals that the topology of the cluster undergoes a discontinuous change (in the space of parameters). This phenomenon is investigated in detail for the cubic case and its connections with the Painlevé I equation. A Zabrodin and P Wiegmann give a detailed account of the so-called Dyson gas, generalizing to arbitrary β the normal matrix-model investigated in their earlier work and, e.g., in the preceding paper of R Teodorescu. They provide detailed information on the $1/N$ expansion, in particular the two next subleading terms which are related to regularized determinants and the generalized Polyakov–Alvarez formula for determinants of unbounded domains ('exterior domains'). The main tools of this study, as in Chekhov's paper, are the so-called *loop* equations, which express the reparametrization invariance of certain path-integrals.

The third part of this issue consists first of two papers, by J Baik *et al* (p8965) and T Suidan (p8977), that are of a probabilistic nature. The work by J Baik *et al* is an intriguing application of the GUE to the modelling of transit times of buses in Cuernaveca, where no a-priori bus-scheduling is implemented, but rather an *ad hoc* system of 'level repulsion' is introduced through observers who communicate the intervals between departures to the successive drivers. T Suidan's paper concerns an application of a theorem of Chattarjee to the universality of the distribution of last-time-passage in percolation models, drastically simplifying other approaches to the problem.

There follow two papers of quite different character, by P J Forrester and N Witte (p8983), and by A Kokotov and D Korotkin (p8997), relating broadly to classical integral systems, and a third, relating to quantum integrable systems, by F Colomo and A G Pronko (p9015). The work by P J Forrester and N S Witte concerns the relationship between the Laguerre ensemble and the Painlevé III and V equations. In order to describe the generating function for the probability of having k eigenvalues at the hard-edge they relate the partition function of the model to the isomonodromic tau-function of Jimbo-Miwa-Ueno and hence to the so-called σ representation of the Painlevé equations. The paper by A Kokotov and D Korotkin is mainly a review of results obtained in recent years by the two authors on the so-called 'Bergman tau-function'. This notion arose in the study of isomonodromic deformations of rational connections with quasi-permutation monodromies but has subsequently appeared in *a priori* unrelated problems like the computation of ζ-regularized determinants of Laplace operators, the so-called G-function of Frobenius manifolds (a particular term in the genus expansion) and the sub-leading term of the large N asymptotic expansions of the partition function of matrix models. Finally the paper by F Colomo and A G Pronko connects certain classical orthogonal polynomials to the statistical mechanics of the six-vertex model (or 'square ice'). It deals with the enumeration of matrices with alternating signs and their refined enumerative properties.

We hope that the reader will find this collection of papers to be a useful and stimulating sample of the very remarkable work that characterizes current research in the rapidly evolving area of random matrices and its relations to integrable systems.

M Bertola and J Harnad

Guest Editors